APPROXIMATE DETERMINATION OF THE FIRST ROOT OF THE CHARACTERISTIC EQUATION IN HEAT CONDUCTION PROBLEMS

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We present an expression for calculating the first root of the characteristic equation in heat conduction problems with boundary conditions of the first and third kinds.

In order to make practical use of the solutions of heat conduction problems for a hollow cylinder or hollow sphere we need to know the roots μ_n of the characteristic equation; in particular, it is important to know the first root μ_1, which determines a temperature change in the body in the regular thermal regime. However in the available handbooks on heat conduction theory [1, 2] the tables of μ_1 that are given are mainly for a solid sphere and cylinder. The manner in which μ_1 is presented adds to the difficulty of analyzing the heating regime in the general case, and the available approximate formulas are usually valid only for a fairly narrow range of variation of the defining parameters (the Biot number, in particular).

We consider the problem of heat conduction for a spherical body (a hollow sphere), one surface of which undergoes convective heat transfer with a mean constant temperature and the other surface is thermally insulated; the initial temperature of the sphere is equal to zero:

$$\frac{1}{(1-K)^2}\cdot\frac{\partial t}{\partial \mathrm{Fo}} = \frac{\partial^2 t}{\partial \rho^2} + \frac{2}{\rho}\cdot\frac{\partial t}{\partial \rho},\quad \mathrm{Fo} > 0. \tag{1}$$

The boundary conditions are

$$\rho = 1,\ \frac{1-K}{\mathrm{Bi}}\cdot\frac{\partial t}{\partial \rho} = t_M - t,\ \rho = K,\ \frac{\partial t}{\partial \rho} = 0. \tag{2}$$

The equation and boundary conditions are written in a form which applies to both the case of heating on the outer surface $0 \le K \le 1$, $K \le \rho \le 1$, as well as to the case of heating on the inner surface $K \ge 1$, $1 \le \rho \le K$.

To solve this problem we made a first approximation as follows:

$$t = \beta(t_M - t_{ins})\frac{1}{\rho}\left(1 - \frac{\rho-1}{K-1}\right)^2 + t_{ins}\ \beta = \frac{\mathrm{Bi}}{\mathrm{Bi}+K+1}. \tag{3}$$

The expression (3) satisfies the boundary conditions on both the heated and the insulated surfaces.

We used L. V. Kantorovich's variational method to determine t_{ins}. After making the transformations we obtained the following expression:

$$\frac{t}{t_M} = 1 + \left[\frac{\beta}{\rho}\left(1 - \frac{\rho-1}{K-1}\right)^2 - 1\right]\exp(-m\,\mathrm{Fo}), \tag{4}$$

$$m = \frac{10\beta\,[3(1+K) - 2\beta]}{10(K^2+K+1) - 5\beta(3+K) + 6\beta^2}. \tag{5}$$

Without going into an analysis of this result, we merely remark that its accuracy can be increased by considering the second and successive approximations. A comparison of expression (4) with the known

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TABLE 1. Comparison of Exact and Approximate Values of μ_1

Bi	0,06	0,1	0,6	0,8	1,0	6,0	8,0	10	20	80	∞
for the infinite plate											
$(\mu_{1approx})_{pl}$	0,2425	0,3111	0,7052	0,7911	0,8606	1,3506	1,4044	1,4365	1,5053	1,5616	1,5811
$(\mu_{1exact})_{pl}$	0,2425	0,3111	0,7051	0,7910	0,8603	1,3496	1,3978	1,4289	1,4961	1,5514	1,5708
δ, %	—	—	0,01	0,01	0,03	0,07	0,47	0,53	0,61	0,66	0,67

Bi	0,1	1,0	2,0	10,0	∞
for the solid sphere					
$(\mu_{1approx})_{sph}$	0,5431	1,5811	2,0412	2,8504	3,1623
$(\mu_{1exact})_{sph}$	0,5423	1,5708	0,0288	2,8363	3,1416
δ, %	0,15	0,67	0,61	0,50	0,66

Table 2. Comparison of the Exact and Approximate Values of μ_1 for the Solid and hollow cylinders

	K=0				K=0,2			
Bi	0,1	1,0	10	∞	0,08	0,8	8,0	80
$(\mu_{1approx})_{cyl}$	0,4271	1,2208	2,1439	2,3717	0,3552	1,02805	1,85925	2,05803
$(\mu_{1exact})_{cyl}$	0,4417	1,2558	2,1795	2,4048	0,36096	1,03412	1,8448	2,0358
δ, %	3,3	2,8	1,6	1,4	1,6	0,6	0,8	1,1

	K=0,4			K=0,6			K=0,8		
Bi	0,06	0,6	6,0	0,08	0,8	20	0,1	1,0	10
$(\mu_{1approx})_{cyl}$	0,28966	0,84707	1,6228	0,31205	0,8869	1,6718	0,32801	0,90764	1,5046
$(\mu_{1exact})_{cyl}$	0,29004	0,84626	1,6175	0,31216	0,8856	1,6584	0,32790	0,90675	1,4986
δ, %	0,1	0,1	0,3	0,1	0,15	0,8	0,1	0,1	0,4

	K=2,0		K=3,0	K=4,0
Bi	1	∞	∞	∞
$(\mu_{1approx})_{cyl}$	0,7192	1,3502	1,2762	1,2121
$(\mu_{1exact})_{cyl}$	0,7057	1,361	1,25	1,175
δ, %	1,9	0,7	2,1	2,9

solutions of this heat conduction problem shows that the quantity m corresponds to the square of the first root of the characteristic equation, i. e.,

$$m = (\mu^2_{1approx})_{sphere} \tag{6a}$$

Thus the expression (5) can be used to determine $\mu_{1approx}$. In the other special cases this expression assumes a simpler form:

for a solid sphere (K = 0):

$$m_0 = (\mu^2_{1approx})_{0\,sphere} = \frac{Bi\,(Bi + 3)}{0.1\,Bi^2 + 0.5\,Bi + 1}, \tag{6b}$$

for an infinite plate (K = 1):

$$m_{plate} = (\mu^2_{1approx})_{sphere} = \frac{Bi\,(Bi + 3)}{0.4\,Bi^2 + 2\,Bi + 3}, \tag{6c}$$

for boundary conditions of the first kind (Bi → ∞):

$$m_{\text{sphere}} = (\mu^2_{1\text{approx}})_{\text{plate}} = \frac{3K+1}{K^2+0,5K+0,1}. \tag{6d}$$

To estimate the accuracy of these expressions we compared the values of $\mu_{1\,\text{approx}}$ in the case of the solid sphere (K = 0) and the infinite plate (K = 1), for various values of Bi, with the exact values $\mu_{1\,\text{exact}}$, given in [1]. The results are shown in Table 1. This analysis shows that the accuracy of the expressions we have obtained for μ_1 in the case of the sphere and the plate is sufficiently high, the maximum error not exceeding 0.67%.

The expressions (5) and (6a)-(6d) also enable us to calculate the values of μ_1 for a long hollow cylinder, namely, $\mu_{1\,\text{cyl}}$.

As the calculations show, the values of $\mu_{1\,\text{cyl}}$, with an accuracy sufficient for applications, obey the relation

$$\mu_{1\text{cyl}}(K,\ \text{Bi}) \approx \frac{1}{2}\,[\mu_{1\text{approx}}(K,\ \text{Bi}) + \mu_{1\text{plate}}(\text{Bi})]. \tag{7}$$

Substituting into Eq. (7) the values $(\mu_{1\,\text{approx}})_{\text{sphere}}$ and $(\mu_{1\,\text{approx}})_{\text{plate}}$, calculated, respectively, from Eqs. (6d) and (6c), we can determine $(\mu_{1\,\text{approx}})_{\text{cyl}}$.

When we compare the values of $(\mu_{1\,\text{approx}})_{\text{cyl}}$, obtained in this way, with the exact values $(\mu_{1\,\text{exact}})_{\text{cyl}}$, we find that, for applied purposes, the error made is completely allowable, although somewhat higher than that for $\mu_{1\,\text{sphere}}$ and $\mu_{1\,\text{plate}}$. The results appear in Table 2. The exact values $(\mu_{1\,\text{exact}})_{\text{cyl}}$ used were as follows: for the solid cylinder (K = 0), we used the value given in [1]; for the hollow cylinder with outer surface heated (K < 1), we used the value given in [4]; finally, for the hollow cylinder with heating on the inner surface, we used the value given in [5].

NOTATION

t	is the temperature;
r, R	are the radii;
$\rho = r/R_{hs}$	is the dimensionless radius;
$K = R_{ins}/R_{hs}$;	
τ	is the time;
$Fo = a\tau/(R_i - R_o)^2$	is the Fourier number;
$Bi = \alpha(R_i - R_o)/\lambda$	is the Biot number;
a	is the thermal diffusivity;
α	is the heat transfer coefficient;
λ	is the thermal conductivity;
μ	is the root of characteristic equation.

Subscripts

hs	is the heated surface;
ins	is the insulated surface;
i and o	are the inner and outer surfaces;
M	is the heated medium.

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